§1. nodal curve : / field, Artin app., / gen. base, structure, $\mathrm{Sing}(f) := \mathrm{ann}(\Omega^2_{X/S})$.

(reduce formal nbhd to $Q(x,y)-c$)

§2. Main Thms : 1 & 2. Improvements,

$Bl_T X$ $\quad$ $Bl_{(S,x)} X$.

$n(T)$ $\quad$ $\Gamma(X)$

§3. $\overline{\mathcal{M}_{g,n}}$.

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§1. Running assumption $f: X \to S$ finite type, Noeth. schemes. (*).

$Sm(f) \subseteq X$ dense.

Defn. A nodal curve $/k$ is an $f: C \to k$ s.t.

(1) C is equi-dim 1;

(2) for each (closed) pt $t \in C \setminus Sm(f)$, $\kappa(t)/k$ is finite sep.

& • $\hat{C}_t \simeq \kappa(t)[[u,v]]/q(u,v)$ for a non-deg. quadratic form. q

(Artin approx.) $\Updownarrow$

• $\exists$ pted étale (C',t'), $(C',t') \to (C,t)$, $(C',t') \to (\mathrm{Spec}(k[x,y]/(xy)), (x,y))$.

Example: $\mathrm{char}(k) \neq 2$, $C = \mathrm{Spec}(k[x,y]/(y^2 - x^2(x+1)))$, exercise to check

$t = (x,y)$

$C' :=$ adjoin $\sqrt{x+1}$, $(y - x\sqrt{x+1})\cdot(y + x\sqrt{x+1})$.

étale around t.

Defn. $f: X \to S$ is called a family of nodal curves if

(0) (*)

(1) flat.

(2). fibers $X_s \to s$ are nodal curves.

Local Structure:

Prop. $(X,x) \longrightarrow (S,s)$ as above.

Then $\underset{\substack{\| \\ (A,m)}}{\hat{\mathcal{O}}_{S,s}} \longrightarrow \underset{\substack{\| \\ B}}{\hat{\mathcal{O}}_{X,x}}$ is

(1) Either $B \simeq A[[u]]$ $\quad (\Leftrightarrow x \in Sm(f))$

(2) or $B \simeq A'[[u,v]]/(Q(u,v) - \underset{\substack{\Downarrow \\ mA'}}{a})$ $\quad (\Leftrightarrow x \notin Sm(f))$.

where $A \to A'$ finite ~~étale~~ corresp to $\kappa(x)/\kappa(s)$

pf: if $x \notin Sm(f)$, then $\kappa(x)/\kappa(s)$ is f. sep.

$\Rightarrow A \longrightarrow A' \longrightarrow B$ (as B is henselian local).

By assumption $B/mB \simeq \kappa(x)[[u,v]]/\bar{q}(u,v)$. non-deg.

Then choose $\tilde{u}, \tilde{v}$ in B & lift $\bar{Q}$ $\Rightarrow$ $B \simeq A'[[u,v]]/(Q(u,v) - \underset{\substack{\Downarrow \\ mA'[[u,v]]}}{g})$.

Exercise: By modifying $\tilde{u}$ & $\tilde{v}$, we can absorb non-const. part of g $\square$.

Thm: If $x \notin Sm(f)$, set $s = f(x)$. Then

$$\exists \quad \begin{tikzcd} & (Y,y) \arrow[dl, "\text{pted ét}"'] \arrow[dr, "\text{pted ét}"] & \\ (X,x) \arrow[d] & & \mathrm{Spec}\left(\mathcal{O}_{\tilde{S}}[u,v]/(uv-a)\right) \arrow[d] \\ (S,s) & & (\tilde{S},\tilde{s}) \arrow[ll, "\text{pted ét.}"'] \end{tikzcd}$$

for some $a \in \mathcal{O}(\tilde{S})$. $\square$

pf: Artin approx. + the above Prop. + absolute Noeth. approx., see [OCBY].

Cor : X/S family of nodal curves.

(1) $\Omega^2_{X/S}$ is locally principally gen. ($\Rightarrow$ $\mathrm{ann}(\Omega^2_{X/S}) \subseteq \mathcal{O}_X$ is étale local on X).

(2). $\mathrm{ann}(\Omega^2_{X/S}) = \mathrm{Fit}_1(\Omega^1_{X/S}) \subseteq \mathcal{O}_X$

cutting out $\underbrace{\text{closed subsch.}}_{\mathrm{Sing}(f)}$ supp. exactly on $X \setminus \mathrm{Sm}(f)$.

(3). If $x \notin \mathrm{Sm}(f)$. Then $a \cdot \mathcal{O}^{s.h.}_{S,s}$ is indep. of the choice.

$\sqsubseteq \mathrm{ann}_{\mathcal{O}^{sh}_{S,s}}(\Omega^2_{X^{sh}_{,x}/S^{sh}_{,s}})$.

(4) $\mathrm{Sing}(f) \longrightarrow S$ is unramified.

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§ 2. Improvement statements.

Setup: regular $S \supseteq D = \bigcup_{i=1}^{r} D_i$ SNCD. $f^{-1}(S \setminus D) \xrightarrow{f} S \setminus D$ is smooth.

Thm 1: $\exists$ successive blow-up $\tilde{X} \xrightarrow{\pi} X \xrightarrow{f} S$ s.t.

(1) center $\subseteq \mathrm{Sing}(X)$ (non-reg. locus, $\subseteq \mathrm{Sing}(f)$).

(2) $f \circ \pi$ is a family of nodal curves (w/ same assumption in setup).

(3) $\mathrm{Sing}(\tilde{X}) \subseteq \tilde{X}$ codim ≥ 3.

$\ulcorner$ Local structure : $\forall x \in \mathrm{Sing}(f)$, $s = f(x) \in S$, choose $\boxed{\cdot}$, $V(a) = \sum_{i=1}^{r} n_i \tilde{D}_i$ as divisors.

$\tilde{D}_i := \tilde{S} \times_S D_i$. (it's irred. after shrinking $\tilde{S}$: D_i's are normal, so $\tilde{D}_i$'s are locally irred.)

So locally equation $Q(u,v) - \prod_{i=1}^{r} t_i^{n_i}$.

Now let $T \subseteq \mathrm{Sing}(X)$ be a codim 2 component.

say $f(T) \subseteq D_1$.

Claim: (0) These n_i only depends on x, $n_i(x)$.

pf: $a \cdot \mathcal{O}^{sh}_{S,s}$ indep. of choice.

(1) $T \longrightarrow D_1$ is étale.

pf: it's unramified + dominant (dim'n reason), D_1 is normal (regular) □

(2) $\forall\, x \in T$, $n_1(x)$ is a constant, $n(T)$

pf: can compare w/

$$\begin{tikzcd} \eta_T \arrow[r, squiggly] \arrow[d] & x \arrow[d] \\ \eta_{D_1} \arrow[r, squiggly] & s \end{tikzcd}$$

(3) $n(T) \geq 2$.

pf: $T \subseteq \mathrm{Sing}(f)$. ⌟

pf of Thm 1: $Bl_T X \to X$ satisfies (1) & (2),

codim 2 components of $\mathrm{Sing}(\tilde{X})$ $\downarrow$ is inj.

—— " —— $\mathrm{Sing}(X)$

$\tilde{T} :=$ strict transf. of T.

Either $\tilde{T} \not\subseteq \mathrm{Sing}(\tilde{X})$ or $n(\tilde{T}) = n(T) - 2$.

$(n(T) = 2$ or $3)$. □.

Defn. C/k nodal curve is <u>split</u> if (1) each component is geom. irred + smooth

(2) all nodes ($=\cap$ of 2 components) are k-rat'l.

X/S family of nodal curves is <u>split</u> if $X_s/k(s)$ is split $\forall\, s \in S$.

Thm 1': If X/S is split, then $\exists\ \tilde{X} \longrightarrow X$ w/

(1), (3) as before & (2') $\tilde{X}/S$ split.

pf idea: In this case, we only have to pass to $\mathcal{O}^h_{S,s}$ & local eqn is $uv - \prod t_i^{n_i}$. □

Thm 2: X/S in the setup + split.

Then $\exists$ $\tilde{X} \xrightarrow{\pi} X$ successive blow-up

(1) center $\subseteq \mathrm{Sing}(X)$

(2) $\tilde{X}/S$ split nodal

(3). $\tilde{X}$ is regular.

pf: Apply Thm I', we already have $\mathrm{sing}(X) \subseteq X$ codim ≥ 3.

(regular $\Leftrightarrow$ $|I|=1$).

Local structure: $uv - \prod_{i \in I \subseteq \{1,\cdots,r\}} t_i$ $m \leq r$.

Consider $X_{\eta_{D_i}}$, we see (1) $\mathrm{Sing}(f) = \bigcup_{i=1}^{r} \bigcup_{j=1}^{l_i} T_i^{(j)}$

w/ each $T_i^{(j)} \xrightarrow{\text{open}} D_i$.

(2) $\mathrm{Sing}(X) = \bigcup$ (intersection of 2 $T_i^{(j)}$'s.)

and has pure codim 3. (cut out by u, v, t_{i_1}, t_{i_2}).

Define:

$\Gamma(X)$ = graph w/ vertex $\leftrightarrow T_i^{(j)}$

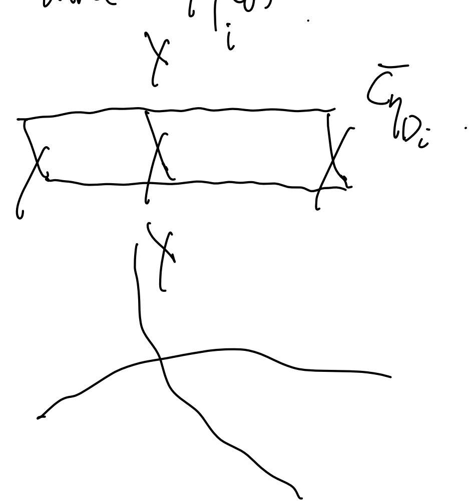
edge $\leftrightarrow$ if 2 $T_i^{(j)}$'s intersect.

Now if $T_i^{(j)}$ intersect w/ another $T_{i'}^{(j')}$ (necessarily $i \neq i'$).

Let $C_{\eta_{D_i}} \subseteq X_{\eta_{D_i}}$ be a comp. passing thru $\eta_{T_i^{(j)}}$.

$\overline{C_{\eta_{D_i}}} \subseteq X$

(picks tangent direction along $T_i^{(j)} \subseteq X_{D_i}$).



(A, m), $s, t \in m$ parameters.

$$\left(A[x,y]/(xy-st),\ m+(x,y)\right) \supseteq \underbrace{\{x=y=st=0\}}_{\mathrm{Sing}(f)}.$$

Blow-up at (s, x).

"$x \neq 0$": $A[s', x, y]/(s - s'\cdot x,\ y - s'\cdot t)$.

$$\cong A[s', x]/(s - s'\cdot x) \supseteq \underbrace{\{s' = x = s = 0\}}_{\mathrm{Sing}(\tilde{f})}$$ above $V(s)$.

"$s \neq 0$": $A[x', x, y]/(x - s\cdot x',\ x'\cdot y - t)$

$$\cong A[x', y]/(x'\cdot y - t) \supseteq \underbrace{\{x' = y = t = 0\}}_{\mathrm{Sing}(\tilde{f})}$$ above $V(t)$.

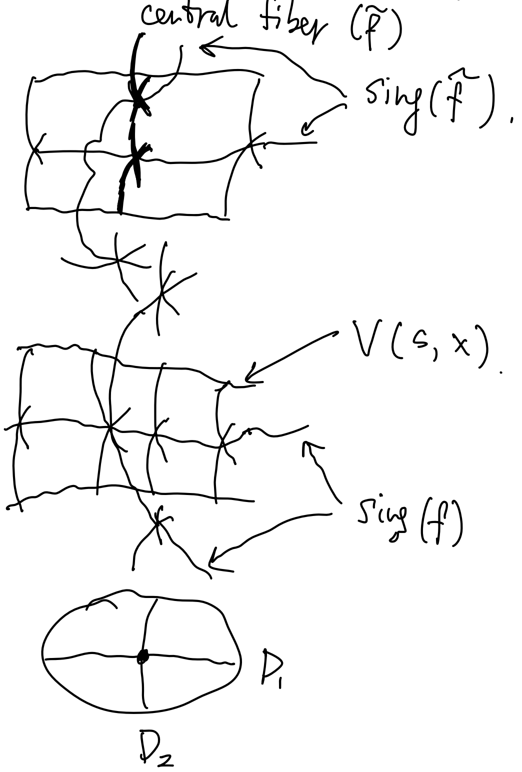
overlap: $A[x', s']/(x'\cdot s' - 1) \supseteq \emptyset = \mathrm{Sing}(\tilde{f})$

Picture:

$$\begin{tikzcd} \tilde{X} \arrow[d, "\pi"] \arrow[dd, bend right, "\tilde{f}"'] \\ X \arrow[d, "f"] \\ S \end{tikzcd}$$

$X =$

$S =$



$$\begin{array}{l} \Gamma(\tilde{X}) \\ \downarrow \\ \Gamma(X) \end{array}$$ same vertices, fewer edges!

§3. $\overline{\mathcal{M}_{g,n}}$. Fix $g,n \in \mathbb{N}$ s.t. $2g-2+n>0$.

Defn. $(C,(\sigma_1,\dots,\sigma_n)\in C^{sm}(k)^n)$ is a genus g, n-marked, stable curve if

(1) nodal + proper.

(2) arithmetic genus $(C)=g$

(3) $\forall$ irred component $C_0 \subseteq C$, if $C_0^\nu \simeq \mathbb{P}^1$ then there are ≥ 3 special pts (preimage of nodes or σ_i)

$\Updownarrow$

(3') $\underline{\mathrm{Aut}}(C/k)$ is finite étale

$\Updownarrow$

(3'') $\omega_{C/k}(\sum_{i=1}^n \sigma_i)$ is ample.

$(X/S,\ \sigma_1,\dots,\sigma_n \in X^{sm}(S))$ is —— " —— .

f.pr. + flat + fiberwise.

(D-M, Knudsen)

Fact: $\exists$ DM-moduli stack $\overline{\mathcal{M}_{g,n}}$ parametrizing these.

(1) smooth & proper / $\mathrm{Spec}(\mathbb{Z})$.

(2) Irreducible, w/ dense open $\mathcal{M}_{g,n}$ (= smooth locus).

(3) Coarse moduli space $\overline{M_{g,n}}$ is a proj. scheme / $\mathbb{Z}$.

(4) $\exists$ irred. projective scheme $M/\mathbb{Z}$ + $M \xrightarrow{\text{finite}} \overline{\mathcal{M}_{g,n}}$.